Combinatorics under Determinacy

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Overview

- Combinatorics
- The Axiom of Determinacy
- Definable Combinatorics

The Simplest Combinatorics: Intuitively

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The Simplest Combinatorics: Intuitively

- The Pigeonhole Principle: “if you have more people than you have beverage types, then at least two people have to have the same beverage.”
- Ramsey’s theorem: “if you have a lot more people than you have beverage types, then there is a large group of people so that every pair pulled from this group has the same combination of beverages”

The Simplest Combinatorics: Formally

- The Pigeonhole Principle: If $m < n \in \mathbb{N}$, X is a set of size n, and $f : X \to m$ is a partition of X into m-pieces, then for some $i < m$, $f^{-1}(i)$ is bigger than 1. (Dirichlet 1834, "Schubfachprinzip")

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- Ramsey's theorem: Fix $n, m, k, l \in \mathbb{N}$. Then there is an $N \in \mathbb{N}$ so that whenever X is a set of size n, and $f : [X]^k \to m$ is a partition of the increasing k-tuples of X into m-pieces, then there is an $A \subseteq X$ so that A has size l and f is constant on $[A]^k$. (Ramsey 1930, [18])

The Coloring Picture

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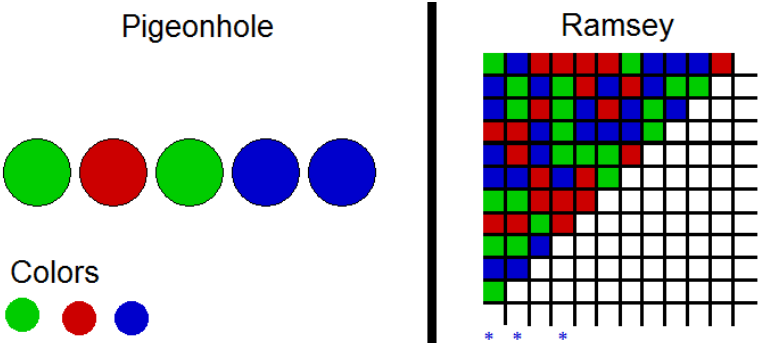

Two Generalizations

There are two ways one might try to generalize these properties.

- Direction 1: add structure to the set being colored and demand that the coloring respects this structure. For example, look at finite graphs and demand that adjacent nodes receive different colors.
- Direction 2: Allow the parameters in the coloring set up to be infinite.

Calibrating Infinite Sizes

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2. Like $\mathbb{N}$, cardinals are well-ordered. Recursive constructions and inductive proofs can be carried out on cardinals.
3. All the finite numbers are represented as cardinals; they form an initial segment of the cardinals.
4. $\aleph_0$ is the first infinite cardinal, it is essentially $\mathbb{N}$. The first uncountable cardinal is $\aleph_1$.
5. If a set X can be well-ordered, then it is in bijection with a unique cardinal κ. We say X has size κ. AC implies every set is in bijection with a unique cardinal.

Calibrating Infinite Sizes

Unlike finite numbers, infinite cardinals can be well-ordered in a variety of ways. These are naturally ordered by order-preserving embeddings and constitute the ordinal numbers. The cardinals and ordinals together form the set theorists number line.

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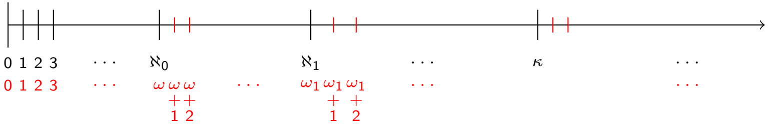

ω is the minimum well-order on $\aleph_0$. It is also essentially $\mathbb{N}$. There are $\aleph_1$-many well-orders on $\aleph_0$. ω_1 is the minimum well-order $\aleph_1$, and there are $\aleph_2$-many well-orders on $\aleph_1$. This pattern continues.

Infinite Combinatorics

For all cardinals we obtain a version of the pigeonhole principle. Suppose κ and λ are cardinals and $\lambda < \kappa$. Suppose X has size κ and $f : X \to \lambda$ is a coloring of X with λ-many colors. Then there is an $\alpha \in \lambda$ so that $f^{-1}(\alpha)$ is bigger than 1.

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The infinite Ramsey theorem is an extension of Ramsey's theorem to all of $\mathbb{N}$. If $m, k < \aleph_0$ and $f : [\aleph_0]^k \to m$, then there is an infinite $A \subseteq \aleph_0$ so that f is constant on $[A]^k$.

Harder Infinite Combinatorics

For infinite cardinals κ, let $[\kappa]^{<\omega}$ be the collection of all increasing finite tuples from κ. Can we get a simultaneous version of Ramsey's theorem for $\aleph_0$: i.e. if $f : [\aleph_0]^{<\omega} \to 2$, is there an infinite $A \subseteq \aleph_0$ so that f is constant on $[A]^{<\omega}$?

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No! Consider $f(\vec{s}) =$ parity of $\mathrm{lh}(\vec{s})$. Let's weaken the question. If $f : [\aleph_0]^{<\omega} \to 2$, is there an infinite $A \subseteq \aleph_0$ so that for each k, f is constant on $[A]^k$?

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No! Consider $f(\vec{s}) = 1$ iff $\min(s) < \mathrm{lh}(\vec{s})$. Let's weaken the question again. Is there a cardinal κ so that whenever $f : [\kappa]^{<\omega} \to 2$, there is an $A \subseteq \kappa$ with size κ so that for each k, f is constant on $[A]^k$?

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- κ is **Rowbottom** if whenever $\lambda < \kappa$ and $f : [\kappa]^{<\omega} \to \lambda$, there is an $A \subseteq \kappa$ with size κ so that when f is restricted to $[A]^{<\omega}$, it's range is countable (Rowbottom 1964, [19]).

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- κ is **Jónsson** if whenever $f : [\kappa]^{<\omega} \to \kappa$, there is an $A \subseteq \kappa$ with size κ so that when f is restricted to $[A]^{<\omega}$, it's range is **not** all of κ (Jónsson 1972, [10]).

Infinite Coloring Properties

Why only allow the size of the set and the number of colors to be infinite? Suppose $f : [\aleph_0]^\omega \to 2$. Must there be an infinite $A \subseteq \aleph_0$ so that f is constant on $[A]^\omega$?

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However, there is a natural topology to put on the $[\aleph_0]^\omega$, and if f corresponds to a Borel set in this topology, then the answer is yes (Galvin-Prikry 1973, [5]). The coloring constructed from the axiom of choice is pathological in much the same way as the Vitali set.

Another Fork in the Road

- Subdirection 1: Embrace the axiom of choice and explore the finite coloring properties under the axiom of choice.
- Subdirection 2: Consider only definable colorings and see what happens when obvious pathologies are avoided.

Extending the Borel Sets

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- Solovay, in 1970 studied an object called $L(\mathbb{R})$ [22]. This is the smallest structure containing $\mathbb{R}$ and closed under all definable operations.
- Unlike the Borel sets, $L(\mathbb{R})$ captures more than just subsets of $\mathbb{R}$, it also captures collections of subsets of $\mathbb{R}$, families of collections of subsets of $\mathbb{R}$, etc...

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- Solovay, in 1970 studied an object called $L(\mathbb{R})$ [22]. This is the smallest structure containing $\mathbb{R}$ and closed under all definable operations.
- Unlike the Borel sets, $L(\mathbb{R})$ captures more than just subsets of $\mathbb{R}$, it also captures collections of subsets of $\mathbb{R}$, families of collections of subsets of $\mathbb{R}$, etc... .
- The properties of Borel sets lift to sets of reals in $L(\mathbb{R})$: they are Lebesgue measurable, have the Baire property, are either countable or in bijection with $\mathbb{R}$, and so on. In fact, a stronger principle which implies all of these is true for $L(\mathbb{R})$.

Games

Let $A \subseteq \mathbb{R}$. The game $\mathcal{G}_A$ is played as follows:

- there are two players, I and II,
- they alternate playing natural numbers,
- this forms an infinite string $\langle n_0, n_1, \cdots \rangle$, which in turn defines a real $x \in \mathbb{R}$,
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I	n_0		n_2		$\cdots$		
II		n_1		n_3		$\cdots$	

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- For player I, this is a function

$$\sigma : \{\langle n_0, n_1, \cdots, n_{2k-1}, n_{2k}\rangle : k \in \mathbb{N} \text{ and } n_0, \cdots, n_{2k} \in \mathbb{N}\} \to \mathbb{N}$$

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  If $y = \langle n_1, n_3, \cdots\rangle$ is II's play in the game, then

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- The situation for player II is similar.

Winning Strategies

A strategy σ for player I is **winning for** $\mathcal{G}_A$ if $\sigma * y \in A$ for every y.
A strategy for player II is **winning for** $\mathcal{G}_A$ if $\tau * y \notin A$ for every A.
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Note:

- If A decides who wins the game after only finitely many moves, then A is determined.
- Only one player can have a winning strategy.
- If A is Borel set, then A is determined (Gale-Stewart 1953, [4]) (D. Martin 1975, [16]).
- Under the axiom of choice, there is a set A which is not determined.

The Axiom of Determinacy

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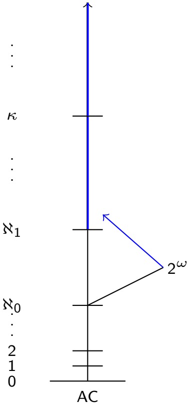
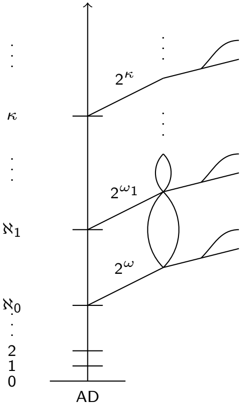
- AD implies that all sets of reals are Lebesgue measurable, have the Baire property, are either countable or in bijection with $\mathbb{R}$, and so on.
- AD contradicts the axiom of choice. In fact, AD implies that there is no well-order on $\mathbb{R}$.
- AD is true for $L(\mathbb{R})$ (Woodin, 1980s). Builds on work of Martin and Steel. For a reference see [13]

Size Without the Axiom of Choice

Without the axiom of choice, the best way to measure size is through injections. The cardinals are no longer a comprehensive list of all possible sizes. Note that 2^ω is in bijection with $\mathbb{R}$.





Finite Coloring Properties Under AD

In settings without the axiom of choice, Θ is used to denote the least cardinal which $\mathbb{R}$ does not surject onto. Under AD, Θ is quite large. In $L(\mathbb{R})$,

- if $\omega < \kappa < \Theta$ is regular, then κ is Ramsey (Steel 1995, [23]),
- if $\omega < \kappa < \Theta$ is regular or is the countable union of sets of smaller cardinality, then κ is Rowbottom, and
- if $\omega < \kappa < \Theta$, then κ is Jónsson (Jackson-Ketchersid-Schlutzenberg-Woodin 2014, [9]).

In fact, this is an exact characterization.

Infinite Coloring Properties Under AD

Building on the work of Mathias from 1976 [17], Shelah and Woodin showed the following in 2002 [20].

Theorem

Suppose $f : [\aleph_0]^\omega \to 2$ is in $L(\mathbb{R})$. Then there is an $A \subseteq \aleph_0$ so that f is constant on $[A]^\omega$.

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Definition

Say κ has the **weak partition property** if whenever $f : [\kappa]^{<\kappa} \to 2$, there is an $A \subseteq \kappa$ so that $|A| = \kappa$ and f is constant on A.
κ has the **strong partition property** if whenever $f : [\kappa]^\kappa \to 2$, there is an $A \subseteq \kappa$ so that $|A| = \kappa$ and f is constant on A.

The Weak and Strong Partition Properties

Theorem (Martin, 1968 [15])

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With his work on descriptions, Steve Jackson has worked to characterize which cardinals have the weak and strong partition properties in $L(\mathbb{R})$.

Combinatorics on Other Sets

$\mathbb{R}$ is the start point for sets which cannot be well-ordered. There are two directions to go from there:

- Stay with linear orders and look at 2^{ω_1}, 2^{ω_2}, etc...
- Go into the cloud and look at quotients of $\mathbb{R}$.

The second direction has the most theoretical support, in the form of descriptive set theory.

Invariant Descriptive Set Theory

The cloud past $\mathbb{R}$ is populated with quotients of $\mathbb{R}$. If E and F are Borel equivalence relations on $\mathbb{R}$, we say $E \leq_B F$ iff there is a map $f : \mathbb{R} \to \mathbb{R}$ so that $xEy \iff f(x)Ff(y)$. This corresponds to $\mathbb{R}/E$ embedding into $\mathbb{R}/F$ in a definable way.

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Suppose that E is a Borel equivalence relation on $\mathbb{R}$. Then either $\mathbb{R}/E$ is countable or $id_{\mathbb{R}} \leq_B E$.

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Define E_0 by xE_0y iff $|x-y| \in \mathbb{Q}$. Note that $E_0 \not\leq_B \mathrm{id}_{\mathbb{R}}$.

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Theorem (Harrington-Kechris-Louveau 1990, [6])

Suppose E is a Borel equivalence relation on $\mathbb{R}$ and $id_{\mathbb{R}} \leq_B E$. Then either $E \leq_B id_{\mathbb{R}}$ or $E_0 \leq_B E$.

Dichotomies Under AD

These dichotomies extend under AD. So, in $L(\mathbb{R})$, if $\aleph_1$ does not embed into $\mathbb{R}/E$, then precisely one of the following is true:

- $\mathbb{R}/E$ is countable,
- $\mathbb{R}/E$ is in bijection with $\mathbb{R}$, or
- $\mathbb{R}/E_0$ embeds into $\mathbb{R}/E$.

Shelah and Harrington proved the first part of this trichotomy for some non-Borel sets in 1980 [7]. Woodin extended this work to all of $L(\mathbb{R})$ in the 90s, and Hjorth proved the last two parts of this trichotomy in 1995 [8]. Caicedo and Ketchersid have recent work extending this to all sets in $L(\mathbb{R})$ [2].

Coloring Properties for Other Sets

When X is just some set, we define $[X]^{<\omega}$ to be the finite subsets of X.

- X is **Ramsey** if whenever $f : [X]^{<\omega} \to 2$, there is an $A \subseteq X$ in bijection with X so that for each k, f is constant on $[A]^k$.
- X is **Jónsson** if whenever $f : [X]^{<\omega} \to X$, there is an $A \subseteq \kappa$ in bijection with X so that when f is restricted to $[A]^{<\omega}$, it's range is not all of X.

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Work from Blass in 1981 [1], Voigt in 1985 [24], and Lefmann in 1987 [14] show that while $\mathbb{R}$ is not Ramsey, there are canonization theorems for $\mathbb{R}$.

Coloring Properties for Pairs of Sets

Definition

Let X and Y be sets. Then

- (X, Y) is **Ramsey** if whenever $f : [X]^{<\omega} \to Y$, there is an $A \subseteq X$ in bijection with X so that for each k, f is constant on $[A]^k$, and
- (X, Y) is **Jónsson** if whenever $f : [X]^{<\omega} \to Y$, there is an $A \subseteq X$ in bijection with X so that when f is restricted to $[A]^{<\omega}$, it's range is not all of Y.

Results for Pairs

Theorem (Jackson-Ketchersid-Schlutzenberg-Woodin, 2014)

Suppose $\omega < \lambda, \kappa < \Theta$ are cardinals. Then in $L(\mathbb{R})$, (κ, λ) is Jónsson.

Theorem (H.-Jackson)

Let X be the set of cardinals between ω and Θ, along with $\mathbb{R}$ and $\mathbb{R}/E_0$. Let $\mathcal{X}$ be the closure of X under $\cup$ and $\times$. Then (A, B) is Jónsson in $L(\mathbb{R})$ for all $A, B \in \mathcal{X}$.

Theorem (H.-Jackson)

$(\mathbb{R}/E_0, \mathbb{R})$ is Ramsey in $L(\mathbb{R})$ and $(\mathbb{R}/E_0, \kappa)$ is Ramsey in $L(\mathbb{R})$ for all cardinals κ.

Thanks For Listening!

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